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LIE IDEALS WITH LEFT GENERALIZED JORDAN TRIPLE DERIVATIONS IN SEMI PRIME RINGS

Dr. C. Jaya subba Reddy
Department of Mathematics,
S.V.University, Tirupati -517502,
Andhra Pradesh, India.

S.Vasantha Kumar
Department of Mathematics,
S.V.University, Tirupati -517502,
Andhra Pradesh, India.

S.Mallikarjuna Rao
Department of Mathematics,
S.V.University, Tirupati -517502,
Andhra Pradesh, India.

Abstract: ***Let R be a 2-torsion free semiprime ring, $L \not\subseteq Z(R)$ a square-closed lie ideal of R, and $F: R \to R$ an additive mapping then F is a left generalized Jordan triple derivation on L associated with a Jordan triple derivation f iff F is a left generalized Jordan derivation on L associated with a Jordan derivation f and let R be a 2-torsion free semiprime ring, $L \not\subseteq Z(R)$ a square-closed lie ideal of R, $F: R \to R$ an additive map such that $F(L) \subseteq L$. If F is a left generalized Jordan triple derivation associated with a Jordan triple derivation f, then F is a left generalized derivation on L associated with a derivation f.***

Keywords: Semiprime ring, Derivation, Left Generalized derivation, Left generalized Jordan derivation, Jordan triple derivation, Centralizer, Lie ideal and Commutator.

INTRODUCTION

Every derivation is a Jordan derivation. In general, the converse is not true. A classical result of Herstein [4] asserts that any Jordan derivation on a prime ring of characteristic different from two is a derivation. A brief proof of the Herstein theorem can be found in Bresar and Vukman [2].Cusack [3] has generalized Herstein theorem to 2-torsion free semiprime ring. Bresar [1] has proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Zalar [9] has proved that any left(right) Jordan centralizer on a semiprime ring is a centralizer. The concept of generalized derivation has been introduced by Bresar in [1] and also the concept of generalized Jordan derivation and generalized Jordan triple derivation have been introduced by Jing and Lu in [7]. Vukman [8] has shown that any generalized Jordan derivation on 2-torsion free semiprime ring is a generalized derivation, and that any generalized Jordan triple derivation on 2-torsion free semiprime ring is a generalized derivation. Hongan.et.all [5] proved every Jordan triple (resp.generalized Jordan triple) derivation on Lie ideal L is a derivation on L(resp.generalized derivation on L).Hongan and Rehman in [6] also extended their results to generalized Jordan triple derivations on lie ideals in semiprime rings. Motivated by above work we extended our results to left generalized Jordan triple derivations on lie ideals in semiprime rings.

Preliminaries: Throughout this Paper, let R will denote an associative ring with center $Z(R)$.U be a non-zero Lie ideal of R. An additive map $d: R \to R$ is a derivation (resp. Jordan derivation) , if $d(xy) = d(x)y + xd(y)$ (resp.$d(x^2) = d(x)x + xd(x)$) ,holds for all $x, y \in R$ .An additive map $f: R \to R$ is right generalized derivation (resp.a right generalized Jordan derivation), if $f(xy) = f(x)y + xd(y)$ (resp.$f(x^2) = f(x)x + xd(x)$) holds for all $x, y \in R$. Left generalized derivation (resp. a left generalized Jordan derivation), if $f(xy) = d(x)y + xf(y)$ (resp.$f(x^2) = d(x)x + xf(x)$) holds for all $x, y \in R$. An additive map $f: R \to R$ is right generalized Jordan triple derivation, if $f(xyx) = f(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in R$. Left generalized Jordan triple derivation $f(xyx) = d(x)yx + xd(y)x + xyf(x)$ holds for all $x, y \in R$. An additive mapping $T: R \to R$ is called a left(right) centralizer in case $T(xy) = T(x)y$ $(T(xy) = xT(y))$,holds for all pairs $x, y \in R$. An additive mapping $T: R \to R$ is called a two-sided centralizer if T is both a left and right centralizer. An additive mapping $T: R \to R$ is called a left(right) Jordan centralizer in case $T(x^2) = T(x)x$ $(T(x^2) = xT(x))$,holds for all $x \in R$. A ring R is called semi prime if $xax = 0$ implies $x = 0$, for all x, a in R. while the symbol $[x, y]$ will denote the commutator $xy - yx$.

Lemma 1: Let R be a 2-torsion free semiprime ring, $L \not\subseteq Z(R)$ a square closed lie ideal of R and 'a' a fixed element of L. If $[x, y]a = 0$, for all $x, y \in L$, then $a \in Z(R) \cap L(= Z(L))$.

Proof: Given that $[x, y]a = 0$. (1)

Substituting ay for y in equation (1) and using (1), we have

$[x, ay]a = 0$

Implies $a[x, y]a + [x, a]ya = 0$

Implies $[x, a]ya = 0$. (2)

So,we get $[x,a]yax=0$. (3)
And substituting yx for y in equation (2), we get
$[x,a]yxa=0$. (4)
Subtracting equation (3) from equation (4), we have
$[x,a]yxa-[x,a]yax=0$
Implies $[x,a]y(xa-ax)=0$
Implies $[x,a]y[x,a]=0$, By [5 ,Corollary 2.1]
We have $[x,a]=0$ for all $x\in L$
i.e., $a\in Z(L)$.

Lemma 2: Let R be a ring, $L\nsubseteq Z(R)$ a square-closed lie ideal of R, and an additive mapping $F:R\to R$ such that $[x,y]F(x)=0$, for all $x,y\in L$ and $F(L)\subseteq L$.

(1) If R is 2-torsion free semiprime, then $F(x)\in Z(L)$, for all $x\in L$.
(2) If R is prime of $char(R)\neq 2$, then $F(x)=0$,for all $x\in L$.

Proof: (1) Suppose that $[x,y]F(x)=0$, for all $x,y\in L$. (5)
Substituting yz for y in equation (5) and using equation (5), we have
$[x,yz]F(x)=0$
Implies $y[x,z]F(x)+[x,y]zF(x)=0$
Implies $[x,y]zF(x)=0$ for all $x,y,z\in L$. (6)
Replacing x by $x+u$ in equation (5) and using equation (5), we have
$[x+u,y]F(x+u)=0$
$[x+u,y]F(x)+[x+u,y]F(u)=0$
$[x,y]F(x)+[u,y]F(x)+[x,y]F(u)+[u,y]F(u)=0$
$[u,y]F(x)+[x,y]F(u)=0$
$[x,y]F(u)=-[u,y]F(x)$,for all $x,y,u\in L$. (7)
Furthermore, substituting $F(u)z[u,y]$ for z in equation (6), we have
$[x,y]F(u)z[u,y]F(x)=0$
Substituting equation (7) in the above equation, we get
$-[u,y]F(x)z[u,y]F(x)=0$
And so, we have $[u,y]F(x)z[u,y]F(x)=0$ for all $z\in L$.
By [5, corollary 2.1(1)], we have
$[u,y]F(x)=0$, for all $u,x,y\in L$, by lemma 1
$F(x)\in Z(L)$, for all $x\in L$. (8)
(2) Since $F(x)\in Z(L)$ for all $x\in L$ by equation (8), $[x,y]LF(x)=0$, for all $x,y\in L$.
By [5, lemma 2.1], we have $F(x)\in Z(L)$ or $[x,y]=0$ for all $x\in L$
Hence, we have $L=\{x\in L\ \backslash F(x)=0\}\cup\{x\in L\backslash[x,y]=0 \text{ for all } y\in L\}$
By Breuer's trick, we conclude that
$L=\{x\in L\ \backslash F(x)=0\}$ or $L=\{x\in L\backslash[x,y]=0 \text{ for all } y\in L\}$
And so, $F(x)=0$ for all $x\in L$.

Lemma 3: Let R be a ring L a lie ideal of R and $F:R\to R$ a derivation on L such that $F(L)\subseteq L$. If $a\in Z(L)$, then $F(a)\in Z(L)$.

Theorem 1: Let R be a 2-torsion free semiprime ring, $L\nsubseteq Z(R)$ a square-closed lie ideal of R, and $F:R\to R$ an additive mapping then the following are equivalent:
(1) F is a left generalized Jordan triple derivation on L associated with a Jordan triple derivation f.
(2) F is a left generalized Jordan derivation on L associated with a Jordan derivation f.

Proof: (1) → (2)
Since F is a left generalized Jordan triple derivation on L.
Therefore, we have $F(xyx)=f(x)yx+xf(y)x+xyF(x)$,for all $x,y\in L$. (9)
In equation (9), we take f as a Jordan triple derivation on L.
Since R is a 2-torsion free semiprime ring, so in view of [5, theorem 2.1] f is a derivation on L.
Now we write $\tau=F-f$.
Then we have $\tau(xyx)=(F-f)(xyx)$
$=F(xyx)-f(xyx)$
$=f(x)yx+xf(y)x+xyF(x)-(f(x)yx+xf(y)x+xyf(x))$
$=f(x)yx+xf(y)x+xyF(x)-f(x)yx-xf(y)x-xyf(x)$
$=xyF(x)-xyf(x)$
$=xy(F(x)-f(x))$
$=xy(F-f)(x)$
$=xy\tau(x)$, for all $x,y\in L$.
In other words, τ is a Jordan triple right centralizer on L.
Since R is a 2-torsion free semiprime ring, one can conclude that τ is a Jordan right centralizer by [5, theorem 3.1].
Hence F is of the form $F=\tau+f$, where f is a derivation and τ is a Jordan right centralizer on L.
Hence F is a left generalized Jordan derivation on L.
(2) → (1), Suppose that $F(x^2)=f(x)x+xF(x)$. (10)
Replacing x by $x+y$ $(y\in L)$ in (10) and using (10), we have
$F((x+y)^2)=\big(f(x+y)\big)(x+y)+(x+y)(F(x+y))$
$F\big((x+y)(x+y)\big)=\big(f(x)+f(y)\big)(x+y)+(x+y)(F(x)+F(y))$
$F(x^2+xy+yx+y^2)=f(x)x+f(x)y+f(y)x+f(y)y+xF(x)+xF(y)+yF(x)+yF(y)$
$F(x^2)+F(xy+yx)+F(y^2)=[f(x)x+xF(x)]+f(x)y+f(y)x+xF(y)+yF(x)+[f(y)y+yF(y)]$
$F(x^2)+F(xy+yx)+F(y^2)=F(x^2)+f(x)y+f(y)x+xF(y)+yF(x)+F(y^2)$
$F(xy+yx)=f(x)y+f(y)x+xF(y)+yF(x)$. (11)

Replacing y by $xy + yx$ in (11) and using (11), we have

$F(x(xy + yx) + (xy + yx)x) = f(x)(xy + yx) + f(xy + yx)x + xF(xy + yx) + (xy + yx)F(x)$

$F(x^2y + xyx + xyx + yx^2) = f(x)xy + f(x)yx + f(xy)x + f(yx)x + xF(xy) + xF(yx) + xyF(x) + yxF(x)$

$F(x^2y + yx^2 + 2xyx) = f(x)xy + f(x)yx + \big(f(x)y + xf(y)\big)x + \big(f(y)x + yf(x)\big)x + x(f(x)y + xF(y)) + x(f(y)x + yF(x)) + xyF(x) + yxF(x)$

$$F(x^2y + yx^2) + 2F(xyx) = f(x)xy + f(x)yx + f(x)yx + xf(y)x + f(y)x^2 + yf(x)x + xf(x)y + x^2F(y) + xf(y)x + xyF(x) + xyF(x) + yxF(x) . \quad (12)$$

On the other hand, substituting x^2 for x in equation (11) and adding $2F(xyx)$ to both sides, we have

$F(x^2y + yx^2) + 2F(xyx) = f(x^2)y + f(y)x^2 + x^2F(y) + yF(x^2) + 2F(xyx)$

$= \big(f(x)x + xf(x)\big)y + f(y)x^2 + x^2F(y) + y\big(f(x)x + xF(x)\big) + 2F(xyx)$

$$= f(x)xy + xf(x)y + f(y)x^2 + x^2F(y) + yf(x)x + yxF(x) + 2F(xyx). \quad (13)$$

From equation (12) and equation (13), we get

$f(x)xy + f(x)yx + f(x)yx + xf(y)x + f(y)x^2 + yf(x)x + xf(x)y + x^2F(y) + xf(y)x + xyF(x) + xyF(x) + yxF(x) = f(x)xy + xf(x)y + f(y)x^2 + x^2F(y) + yf(x)x + yxF(x) + 2F(xyx)$

$f(x)xy + f(x)yx + f(x)yx + xf(y)x + f(y)x^2 + yf(x)x + xf(x)y + x^2F(y) + xf(y)x + xyF(x) + xyF(x) + yxF(x) - \big(f(x)xy + xf(x)y + f(y)x^2 + x^2F(y) + yf(x)x + yxF(x) + 2F(xyx)\big) = 0$

$f(x)xy + f(x)yx + f(x)yx + xf(y)x + f(y)x^2 + yf(x)x + xf(x)y + x^2F(y) + xf(y)x + xyF(x) + xyF(x) + yxF(x) - f(x)xy - xf(x)y - f(y)x^2 - x^2F(y) - yf(x)x - yxF(x) - 2F(xyx) = 0$

$2f(x)yx + 2xf(y)x + 2xyF(x) - 2F(xyx) = 0$

$2(f(x)yx + xf(y)x + xyF(x) - F(xyx)) = 0$

Since R be 2-torsion free semi prime ring then

$f(x)yx + xf(y)x + xyF(x) - F(xyx) = 0$

$f(x)yx + xf(y)x + xyF(x) = F(xyx)$ (or)

$F(xyx) = f(x)yx + xf(y)x + xyF(x)$

Since f is a derivation, f is a Jordan triple derivation and F is a generalized left Jordan triple derivation on associated with a Jordan triple derivation f. ☐

Theorem 2: Let R be a 2-torsion free semiprime ring, $L \nsubseteq Z(R)$ a square-closed lie ideal of R, $F: R \to R$ an additive map such that $F(L) \subseteq L$. If F is a left generalized Jordan triple derivation associated with a Jordan triple derivation f, then F is a left generalized derivation on L associated with a derivation f.

Proof: Suppose that there exists a Jordan triple derivation f on L such that

$$F(xyx) = f(x)yx + xf(y)x + xyF(x) , \ x \in L. \quad (14)$$

Since R is a 2-torsion free ring, f is a derivation by [5, theorem 2.1]

Substituting $x + z$ $(z \in L)$ for x in equation (14) and using (14), we have

$F\big((x + z)y(x + z)\big) = f(x + z)y(x + z) + (x + z)f(y)(x + z) + (x + z)yF(x + z)$

$F(xyx + xyz + zyx + zyz) = \big(f(x) + f(z)\big)(yx + yz) + \big(xf(y) + zf(y)\big)(x + z) + (xy + zy)\big(F(x) + F(z)\big)$

$F(xyx) + F(xyz + zyx) + F(zyz) = f(x)yx + f(x)yz + f(z)yx + f(z)yz + xf(y)x + xf(y)z + zf(y)x + zf(y)z + xyF(x) + zyF(x) + xyF(z) + zyF(z)$

$= [f(x)yx + xf(y)x + xyF(x)] + f(x)yz + f(z)yx + xf(y)z + zf(y)x + zyF(x) + xyF(z) + [f(z)yz + zf(y)z + zyF(z)]$

$F(xyx) + F(xyz + zyx) + F(zyz) = F(xyx) + f(x)yz + f(z)yx + xf(y)z + zf(y)x + zyF(x) + xyF(z) + F(zyz)$

$$F(xyz + zyx) = f(x)yz + f(z)yx + xf(y)z + zf(y)x + xyF(z) + zyF(x). \quad (15)$$

Now we set $A = F(xyzyx + yxzxy)$ and we shall compute it in two different ways using equation (14), we have

$A = F(x(yzy)x + y(xzx)y)$

$= F(x(yzy)x) + F(y(xzx)y)$

$= f(x)yzyx + xf(yzy)x + xyzyF(x) + f(y)xzxy + yf(xzx)y + yxzxF(y)$

$= f(x)yzyx + x[f(y)yz + yf(z)y + yzf(y)]x + xyzyF(x) + f(y)xzxy + y[f(x)zx + xf(z)x + xzf(x)]y + yxzxF(y)$

$$= f(x)yzyx + xf(y)yzx + xyf(z)yx + xyzf(y)x + xyzyF(x) + f(y)xzxy + yf(x)zxy + yxf(z)xy + yxzf(x)y + yxzxF(y). \quad (16)$$

Using (15), $A = F((xy)z(yx) + (yx)z(xy))$

$= f(xy)zyx + f(yx)zxy + xyf(z)yx + yxf(z)xy + xyzF(yx) + yxzF(xy)$

$= [f(x)y + xf(y)]zyx + [f(y)x + yf(x)]zxy + xyf(z)yx + yxf(z)xy + xyzF(yx) + yxzF(xy)$

$$= f(x)yzyx + xf(y)zyx + f(y)xzxy + yf(x)zxy + xyf(z)yx + yxf(z)xy + xyzF(yx) + yxzF(xy). \quad (17)$$

From equation (16) and equation (17), we get

$f(x)yzyx + xf(y)zyx + f(y)xzxy + yf(x)zxy + xyf(z)yx + yxf(z)xy + xyzF(yx) + yxzF(xy) = f(x)yzyx + xf(y)yzx + xyf(z)yx + xyzf(y)x + xyzyF(x) + f(y)xzxy + yf(x)zxy + yxf(z)xy + yxzf(x)y + yxzxF(y)$

$f(x)yzyx + xf(y)zyx + f(y)xzxy + yf(x)zxy + xyf(z)yx + yxf(z)xy + xyzF(yx) + yxzF(xy) - [f(x)yzyx + xf(y)yzx + xyf(z)yx + xyzf(y)x + xyzyF(x) + f(y)xzxy + yf(x)zxy + yxf(z)xy + yxzf(x)y + yxzxF(y)] = 0$

$xyzF(yx) - xyzf(y)x - xyzyF(x) + yxzF(xy) - yxzf(x)y - yxzxF(y) = 0$
$xyz[F(yx) - f(y)x - yF(x)] + yxz[F(xy) - f(x)y - xF(y)] = 0.$ (18)
Now putting $\alpha(x,y) = F(xy) - f(x)y - xF(y)$.(19)
Substituting equation (19) in equation (18), we have
$xyz\alpha(y,x) + yxz\alpha(x,y) = 0$, for all $x,y,z \in L$. (20)
By the way, F is a generalized left derivation on L associated with a derivation f by theorem 1, and so
$F(x^2) = f(x)x + xF(x)$, for all $x \in L$. (21)
Substituting $x + y$ for x in equation (21), we have
$F((x+y)^2) = \big(f(x+y)\big)(x+y) + (x+y)(F(x+y))$
$F\big((x+y)(x+y)\big) = \big(f(x)+f(y)\big)(x+y) + (x+y)(F(x)+F(y))$
$F(x^2) + F(xy+yx) + F(y^2) = F(x^2) + f(x)y + f(y)x + xF(y) + yF(x) + F(y^2)$
$F(xy+yx) = f(x)y + f(y)x + xF(y) + yF(x)$, for all $x,y \in L$. (22)
$F(xy) + F(yx) = f(x)y + f(y)x + xF(y) + yF(x)$
$F(xy) - f(x)y - xF(y) = -F(yx) + f(y)x + yF(x)$
$F(xy) - f(x)y - xF(y) = -[F(yx) - f(y)x - yF(x)]$
Now using equation (19), we have
$\alpha(x,y) = -\alpha(y,x)$, for all $,y \in L$. (23)
Substituting equation (23) in equation (20), we have
$-xyz\alpha(x,y) + yxz\alpha(x,y) = 0$
$(yx - xy)z\alpha(x,y) = 0$
$[x,y]z\alpha(x,y) = 0$, for all $x,y,z \in L$.
By [5, lemma 2.4], we get $[u,v]z\alpha(x,y) = 0$, for all $x,y,z,u,v \in L$ and
We have $[u,v]\alpha(x,y) = 0$ for all $x,y,u,v \in L$ by [5 corollary 2.1(3)] and
So, we get $\alpha(x,y) \in Z(L)$ by lemma 1
Now, we put $\alpha(x,y) = \alpha$ and $B = F(xy\alpha + \alpha xy)$, we will compute that in two different ways.
Using equation (22), we have $B = F(xy\alpha + \alpha xy)$
$= f(xy)\alpha + f(\alpha)xy + xyF(\alpha) + \alpha F(xy)$
$= [f(x)y + xf(y)]\alpha + f(\alpha)xy + xyF(\alpha) + \alpha F(xy)$
$= f(x)y\alpha + xf(y)\alpha + f(\alpha)xy + xyF(\alpha) + \alpha F(xy)$. (24)
Because of $\alpha \in Z(L)$, using equation (15), we have
$B = F(xy\alpha + \alpha xy)$
$= f(x)y\alpha + f(\alpha)xy + xf(y)\alpha + \alpha f(x)y + xyF(\alpha) + \alpha xF(y)$. (25)
From equation (24) and equation (25), we have
$f(x)y\alpha + xf(y)\alpha + f(\alpha)xy + xyF(\alpha) + \alpha F(xy) = f(x)y\alpha + f(\alpha)xy + xf(y)\alpha + \alpha f(x)y + xyF(\alpha) + \alpha xF(y)$
$f(x)y\alpha + xf(y)\alpha + f(\alpha)xy + xyF(\alpha) + \alpha F(xy) - f(x)y\alpha - f(\alpha)xy - xf(y)\alpha - \alpha f(x)y - xyF(\alpha) - \alpha xF(y) = 0$
$\alpha F(xy) - \alpha f(x)y - \alpha xF(y) = 0$
$\alpha\{F(xy) - f(x)y - xF(y)\} = 0$
Using equation (19), we have
$\alpha \times \alpha = 0$
$\alpha^2 = 0$.
And so, we have $\alpha L \alpha = \{0\}$.
Since R is semi prime, $\alpha = 0$.
That is,$F(xy) = f(x)y + xF(y)$, for all $x,y \in L$.
By Theorems 1 and 2 we obtain the following result which explains the relationships of left generalized Jordan triple derivations, left generalized derivations and left generalized Jordan derivations.

Corollary 1: Let R be a 2 – torsion free semiprime ring ,$L \nsubseteq Z(R)$ a square-closed lie ideal of R and let $F: R \to R$ be an additive map such that $F(L) \subseteq L$,then the followings are equivalent:

(1) F is a left generalized Jordan triple derivation on L associated with a Jordan triple derivation f .
(2) F is a left generalized derivation on L associated with a derivation f.
(3) F is a left generalized Jordan derivation on L associated with a derivation f. □

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